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ON WEAKLY BALANCED GAMES AND DUALITY THEORY

by

A. Charnes
K. O. Kortanek*
Mark Eisner*

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*Cornell University

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A. Charnes, Director

1. Introduction

The previous papers [1], [2], and [3] were concerned with solution concepts for finite (n-person) games. [1] and [2] developed a new class of solution concepts, termed <u>nuclei</u>, which were determined by a family of mathematical programming problems involving conditions on the excesses of coalition values with respect to payoff vectors. These concepts included convex nuclei, convex separable nuclei, the special case of quadratic nuclei, and Schmeidler's [4] nucleolus. An earlier development [3] dealt with the solution concept of the core of an n-person game, using the duality theory of linear programming to characterize the core via Shapley's [5] minimal balanced collections and to answer, in the affirmative, a conjecture by Shapley on the sharpness of proper minimal balanced collections. In that paper, a proper operator $M(\cdot)$, defined on coalitions, was introduced to characterize the redundancy of certain coalition inequalities. Roughly speaking, M associates with each coalition the best weighted value among all collections which are balanced with respect to the argument playing the role of grand coalition. D. Schmeidler [6] has defined a game with an arbitrary set of players, and has extended the solution concept of the core and the notion of a balanced game to this case. He has shown that an infinite game for which the range of values of the coalitions is non-negative and bounded has non-empty core if and only if it is balanced. His proof, based on arguments usually used to prove the Hahn-Banach theorem, extends the Charnes-Kortanek M-operator as defined in [3] to this situation.

In this paper we define the notion of a "weakly balanced game" under very general conditions involving no topology whatever. Using the M-operator, we further extend the work of Schmeidler by establishing duality results for a pair of (possibly) infinite dimensional linear programming problems arising from a generalized game. A necessary and sufficient condition is given in order that a separating hyperplane argument can be employed to prove the existence of a candidate core member for a weakly balanced game. This candidate is shown to be in the core if and only if the game is balanced. No use is made of topological ideas, but conditions are given under which the core member takes on values in a bounded set.

Analogous to results in the n-player case, we use the Charnes-Kortanek M-operator to characterize the redundancy of certain coalition values in restricting core membership.

2. Definitions: Generalized Games, Weakly Balanced Games, Outcome and Core of a Generalized Game.

We consider an arbitrary linear vector space V. A subset χ of V, called the set of coalitions and an element X_0 in χ, called the grand coalition, are specified such that the following properties hold:

A. χ spans V; that is, each member of V can be written as a linear combination of finitely many members of χ, and

B. There exists a $P_0 \in V$ such that for each X in χ, there is a finite subset

$\{X_1, X_2, \ldots, X_n\}$ contained in χ, non-negative numbers $\eta_1, \eta_2, \ldots, \eta_n$, and $\eta^* > 0$ such that

$$\sum_{i=1}^{n} \eta_i X_i + \eta^* X = P_0.$$

Property B can be paraphrased as follows: There is a vector P_0 such that each coalition can be incorporated in an expression of P_0 as a weighted sum of coalitions, the weights being positive. If an ordering on V is induced by the cone spanned by χ, property B becomes $X \in \chi \to P_0 \geq \eta^* X$ for some $\eta^* \geq 0$. Note that B is satisfied if $X_0 - X$ is in $\chi, \forall X \in \chi$ since P_0 may be chosen to be X_0. Given a set χ such that A is not satisfied, it will always be possible in this context to restrict attention to the space spanned by χ, so that with this understanding A can always be assumed to hold. In addition to χ and X_0, an arbitrary function v from χ to the real numbers, called the payoff function, is given.[1] The <u>triple</u> $(\chi, X_0; v)$ is called a generalized game.

<u>Example 1</u>:

Let $\sum_1$ be a field of subsets of an arbitrary set S, let χ_1 be the set of characteristic functions of members of $\sum_1$ and let X_0 be the characteristic function of S. Let v_1 be a bounded, non-negative function on χ_1 with v_1 equal to zero on the characteristic function of the empty set and $v_1(X_0)$ positive. $(\chi_1, X_0; v_1)$ is Schmeidler's [6] formulation of a game with infinitely many players, and is a generalized game.

<u>Example 2</u>:

Let $\sum_2$ be a collection of subsets of an arbitrary set S such that if $A \in \sum_2$ then $S - A \in \sum_2$. Let X_0 and χ_2 be defined from $\sum_2$ as

1. v is often called the characteristic function of the game, but this term is reserved for its more usual meaning in subsequent examples, while the term payoff is sometimes used for what we will designate as an outcome. Our usage conforms to Schmeidler's [6].

in example 1, and let v_2 be any real valued function on χ. Then $(\chi_2, X_0; v_2)$ is a generalized game.

Example 3:

Let S be an arbitrary set, and $\chi = [0, 1]^S$ = the set of all functions from S into the interval $[0, 1]$. Let v be an arbitrary real valued function on χ and $X_0 \in [0, 1]^S$. $(\chi, X_0; v)$ is a generalized game, if $X_0(S) = 1$ for all $s \in S$ then the value of a coalition at s might represent the probability that s participates in that coalition.

In all three examples, an appropriate P_0 is the characteristic function of S.

A generalized game is called weakly balanced if

$$\sup \left\{ \sum_{\alpha \in A} \eta_\alpha v(X_\alpha) \,\middle|\, \sum_{\alpha \in A} \eta_\alpha = P_0,\ \eta_\alpha \geq 0,\ A \text{ ranges over all finite sets indexing members of } \chi \right\}$$

is finite, in which case the sup is denoted p_0.

Let $$\sup \left\{ \sum_{\alpha \in A} \eta_\alpha v(X_\alpha) \,\middle|\, \sum_{\alpha \in A} \eta_\alpha X_\alpha = X_0,\ \eta_\alpha \geq 0,\ A \text{ ranges over all finite sets indexing members of } \chi \right\}.$$

The sup is well-defined, since $X_0 \in \chi$ so the set includes $v(X_0)$ at least.

In many cases it is possible to choose $P_0 = X_0$, so that $v_0 = p_0 < \infty$. But even when this is not possible, the finiteness of v_0 is a consequence of the finiteness of p_0.

This follows from the fact that $X_0 \in \chi$ and property B which allows us to write

$$P_o = \sum_{i=1}^{n} \eta_i X_i + \eta^* X_o, \quad \eta_i, \eta^* \geq 0, \quad X_i \in \chi.$$

For any expression

$$X_o = \sum_{j=1}^{m} \eta_j X_j \quad X_o, \; X_j \in \chi, \; \eta_j \geq 0,$$

by substitution $P_o = \sum_{i=1}^{n} \eta_i X_i + \eta^* \sum_{j=1}^{n} \eta_j X_j$, and

hence $p_o \geq \sum_{i=1}^{n} \eta_i v(X_i) + \eta^* \sum_{j=1}^{n} \eta_j v(X_j)$.

By letting the expression of X_o range over all those which are possible, we have

$$\infty > p_o \geq \sum_{i=1}^{n} \eta_i v(X_i) + \eta^* v_o, \quad \text{so } v_o < \infty .$$

The conditions for weak boundedness are essentially conditions on the function v, which cannot be chosen arbitrarily for a weakly balanced game. Nonetheless, v need not be bounded, as the following simple example shows:

Let V be the real line, $\chi = \{X_{N-1} | X_{N-1} = N$ for N a positive integer$\}$ and $P_o = X_o = 1$. Let $v(X_{N-1}) = (-1)^N N$ for $X_{N-1} \in \chi$, and $v(X) = 0$ otherwise. To satisfy property B, choose $\eta^*_{N-1} = \frac{1}{N} > 0$ for $X_{N-1} \in \chi$, so that $\eta^*_{N-1} X_{N-1} = P_o$. Clearly $v_o = p_o = 1$ despite the fact that v is unbounded above and below.

Note that in Example 1, the value of v_o and p_o are unchanged if the equality is replaced by $\leq$. For, in this example, the positive orthant in the space V coincides with the convex cone determined by χ, so if $\sum_{j=1}^{n} \eta_j X_j \leq X_o$ with $\eta_j \geq 0$ and $X_j \in \chi$ then

$$X_o - \sum_{j=1}^{n} \eta_j X_j = \sum_{i=1}^{m} \nu_i X_i$$

for some characteristic vectors $X_i \in \chi$ and $\nu_i \geq 0$.

Since in this example v is a non-negative function,

$$\sum_{j=1}^{n} \eta_j \, v(X_j) + \sum_{i=1}^{m} \nu_i \, v(X_i) \geq \sum_{j=1}^{n} \eta_j \, v(X_j)$$

while $\sum_{j=1}^{n} \eta_j X_j + \sum \nu_i X_i = X_0$ with $\eta_j, \nu_i \geq 0$.

Observe also that all finite games are weakly balanced (see Proposition 4 of [3].)

An outcome of a generalized game is a linear functional, λ, on V such that $\lambda(X_0) = v(X_0)$. An outcome is said to be in the core of the game if for each $X \in \chi$, $\lambda(X) \geq v(X)$.

We will be concerned with deriving conditions for core membership of a weakly balanced generalized game.

3. Formulation as Dual Programs

Henceforth we consider the weakly balanced game $(\chi, X_0; v)$.

Consider the following pair of linear programming problems.

(I)

$$\inf \lambda(X_0)$$

$$\text{s.t.} \quad \lambda(X) \geq v(X)$$

$$X \in \chi$$

λ is a linear functional on V

(II)

$$\sup_{A} \sum_{\alpha \in A} \eta_\alpha \, v(X_\alpha)$$

$$\text{s.t.} \quad \sum_{\alpha \in A} \eta_\alpha X_\alpha = X_0$$

$$\eta_\alpha \geq 0$$

where A ranges over all possible finite index sets for members of χ.

For a weakly balanced game, problem II has a finite supremum, v_o. If a functional λ is I-feasible, and if $\lambda(X_o) = v(X_o)$, then λ is an outcome in the core of the game $(\chi, X_o; v)$.

Proposition 1:

Let (χ, X_o, v) be a weakly balanced generalized game. A necessary condition that its core be non-empty is that $v(X_o) = v_o$.

Proof:

Let λ be I-feasible, $\eta = (\eta_{\alpha_1}, \ldots, \eta_{\alpha_n})$ be II-feasible. Then $\lambda(X_o) = \lambda(\sum_{i=1}^{n} \eta_{\alpha_i} X_{\alpha_i}) = \sum_{i=1}^{n} \eta_{\alpha_i} \lambda(X_{\alpha_i}) \geq \sum_{i=1}^{n} \eta_{\alpha_i} v(X_{\alpha_i})$, by the linearity of λ and the non-negativity of η. Hence $\lambda(X_o) \geq v_o$. But if λ is in the core, $v(X_o) = \lambda(X_o) \geq v_o$, and since $X_o \in \chi$, it follows that $v_o \geq v(X_o)$, so that $v_o = v(X_o)$.

When $v_o = v(X_o)$, the game is said to be balanced. In the next section it is shown that it is sufficient that the game be balanced in order that the core be non-empty.

4. Duality Theory for Weakly Balanced Games

Given a subset $\psi \subset \chi$, the operator $M_\psi: V \to [-\infty, \infty]$, (the extended real line), is defined as follows.[1]

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1. Here M_ψ is an extension of the M-operator for finite games defined by Charnes-Kortanek [3], and closely related to Schmeidler's operator [6]. The M-operator in [3] is given in the present notation as $M_\psi(X)$, where $\psi = \chi - \{X\}$.

$M_\psi(X) = \sup\{\sum_{\gamma\in G} \eta_\gamma\, v(Y_\gamma) \mid \sum_{\gamma\in G} \eta_\gamma Y_\gamma = X,\ \eta_\gamma \geq 0,$ G ranges over all finite index sets of members $Y_\gamma \in \psi\}$ if some appropriate G exists and the sup is finite; otherwise $M_\psi(X) = -\infty$ if no G exists, and $M_\psi(X) = +\infty$ if the set is not bounded above. Note that $M_\chi(X_o) = v_o$.

The following properties of M_ψ are easily verified:

(i) $M_\psi(X_1 + X_2) \geq M_\psi(X_1) + M_\psi(X_2)$ if at least one term on the right is finite.

(ii) If $\alpha > 0$, then $M_\psi(\alpha Y) = \alpha M_\psi(Y)$.

(iii) Restricted to any domain for which $M_\psi(X) > -\infty$, M_ψ is concave.

(iv) Either $M_\psi(0) = 0$ or $M_\psi(0) > 0$. In the latter case if there exists $X \in V$ such that $M_\psi(X) > -\infty$, then $M_\psi(X) = M_\psi(X + a\cdot 0) \geq M_\psi(X) + aM_\psi(0)$ for any $a > 0$, which implies $M_\psi(X) = +\infty$. (If $\sum_{i=1}^{n} \eta_i X_i = 0$, $\eta_i > 0$, $X_i \in \psi$, then the vector η with these η_i's as non-zero terms determines an infinite ray).

To simplify notation, for the remainder of this section, $M_\chi(\cdot)$ will be abbreviated $M(\cdot)$. Note that $M_\chi(X) \geq v(X)$ for $X \in \chi$.

Let $K = \{X \in V \mid M(X) > -\infty$ and $M(P_o - \eta^* X) > -\infty$ for some $\eta^* > 0\}$. The following properties of K are easily proved:

i) $P_o \in K$, since $M(P_o) = p_o > -\infty$ and $M(P_o - \eta^* P_o) > -\infty$ for $\eta^* = 1$, say.

ii) $\chi \subset K$, since property B of the definition is equivalent to $M(P_o - \eta^* X) > -\infty$ for some $\eta^* > 0$.

iii) If $X \varepsilon K$, $M(X) \leq \frac{1}{\eta^*} M(P_o) - \frac{1}{\eta^*} M(P_o - \eta^* X) < \infty$.

Let $\hat{K} = \{(X, z) | X \varepsilon K, z \varepsilon R, M(X) - z > 0\}$, where R is the real line, and let $B = \{(c X_o, |c\, v_o)\ c \geq 0\}$.

Lemma 2 $\hat{K}$ convex, B convex, and $\hat{K} \cap B = \emptyset$

Proof:

Suppose the points (X_1, z_1) and (X_2, z_2) are in $\hat{K}$. Let t be arbitrary in $0 \leq t \leq 1$. We show first that $X_3 = tX_1 + (1 - t)X_2$ is in the set K. By the definition of K, there is an η_i^* such that $M(P_o - \eta_i^* X_i) > -\infty$, $i = 1, 2$. Suppose (without loss of generality) that $\eta_1^* \leq \eta_2^*$, or $\eta_2^* - \eta_1^* \geq 0$. Then $M(P_o - \eta_1^* X_3)$

$$= M(P_o - \eta_1^* tX_1 - \eta_1^*(1 - t)X_2)$$

$$= M((1 - t)P_o + t\, P_o - t\eta_1^* X_1 - (1 - t)(\eta_2^* X_2 - (\eta_2^* - \eta_1^*)X_2)$$

$$= M(t\, P_o - \eta_1^* X_1) + (1 - t)(P_o - \eta_2^* X_2) + (1 - t)(\eta_2^* - \eta_1^*)X_2)$$

$$\geq tM(P_o - \eta_1^* X_1) + (1 - t)\, M(P_o - \eta_2^* X_2) + (1 - t)(\eta_2^* - \eta_1^*)\, M(X_2) > -\infty$$

Also $M(X_3) \geq t\, M(X_1) + (1 - t)\, M(X_2) > -\infty$, so X_3 is in K. Note that we have thereby shown that K is convex.

To show $\hat{K}$ is convex it remains to show that

$M(X_3) - z_3 > 0$, where $z_3 = tz_1 + (1 - t)z_2$. But

$M(X_3) - z_3 \geq t(M(X_1) - z_1) + (1 - t)(M(X_2) - z_2) > 0$, so the convexity of $\hat{K}$ is proved. B is clearly convex, and since

$M(c\,X_o) - c\,v_o = 0$, fo $c \geq 0$, B · . . = ∅ Q.E.D.

A subset of a linear space is called radial[1] at a point X if for each vector of $Y \in V$ there is a $T > 0$ for which $0 < t \leq T$ implies $X + t\,Y$ is in the subset. We wish to demonstrate the existence of a λ feasible for problem I such that $\lambda(X_o) = v_o$. This will depend on whether the set $\hat{K}$ is radial at some point.

<u>Proposition 3</u>.

For a weakly balanced generalized game, $\hat{K}$ is radial at

$$\left(\frac{P_o}{2}, \frac{(P_o - 1)}{2} \right).$$

<u>Proof</u>:

Suppos^ $(Y, z) \in V \times R$ is gi^^n. We show first that $\frac{P_o}{2} + tY$ is in K i.. t small en~igh. Since · spa·s V, by reorc..ring he indices such that $n_i \geq 0$ for $1 \leq i \leq k$ and $n_i < 0$ for $k + 1 \leq i \leq n$ we can write

$$Y = \sum_{i=1}^{n} n_i\,X_i = \sum_{i=1}^{k} \frac{|n_i|}{n_i^*}\,n_i^*\,X_i - \sum_{i=k+1}^{n} \frac{|n_i|}{n_i^*}\,n_i^*\,X_i . \qquad X_i \in X .$$

Here each $n_i^* > 0$ is such that $M(P_o - n_i^*\,X_i) > -\infty$; these constants exist by prcnerty B of the definit on of a generalized game. For each i, let $\frac{|n_i|}{n_i^*} = v_i > 0$.

Let $T_1 = (2 \sum_{i=1}^{n} v_i)^{-1} > 0$, and suppose $0 < t \leq T$.

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1. See [7] Chapter 1.

To show $\frac{P_o}{2} + tY \in K$, it sufficies to show that $M(\frac{P_o}{2} + tY) > -\infty$ and that

$$M(P_o - (\frac{P_o}{2} + tY)) = M(\frac{P_o}{2} - tY) > -\infty.$$ These results follows from the following relations.

$$M(\frac{P_o}{2} + tY) = M(\frac{P_o}{2} + t(\sum_{i=1}^{k} \nu_i + \sum_{i=k+1}^{n} \nu_i - \sum_{i=1}^{n} \nu_i)P_o$$

$$+ t(\sum_{i=1}^{k} \nu_i \eta_i^* X_i - \sum_{i=k+1}^{n} \nu_i \eta_i^* X_i))$$

$$= M((\frac{1}{2} - t\sum_{i=1}^{n} \nu_i)P_o + t\sum_{i=1}^{k} \nu_i (P_o + \eta_i^* X_i) + t\sum_{i=k+1}^{n} \nu_i (P_o - \eta_i^* X_i))$$

$$\geq (\frac{1}{2} - t\sum_{i=1}^{n} \nu_i) M(P_o) \quad t\sum_{i=1}^{k} \nu_i M(P_o + \eta_i^* X_i) \cdot t\sum_{i=k+1}^{n} \nu_i M(P_o - \eta_i^* X_i) > -\infty$$

since $M(P_o \pm \eta_i^* X_i) > -\infty$ for all i and all coefficients are non-negative.

Similarly

$$M(\frac{P_o}{2} - tY) = M((\frac{1}{2} - t\sum_{i=1}^{n} \nu_i)P_o + t\sum_{i=1}^{k} \nu(P_o - \eta_i^* X_i)$$

$$+ t\sum_{i=k+1}^{n} \nu_i (P_o + \eta_i^* X_i)) > -\infty,$$

so $\frac{P_o}{2} + tY \in K$.

Now let $T_2 = \min(T_1, \frac{T_1}{2} / |M(\frac{P_o}{2} + T_1 Y) - \frac{P_o}{2} - T_1 z|) > 0$,

since $\frac{P_o}{2} + T_1 Y \in K$ and hence $M(\frac{P_o}{2} + T_1 Y) < \infty$.

We will show that

$$M(\frac{P_o}{2} + tY) > \frac{P_o}{2} - \frac{1}{2} + tz \quad \text{for } 0 \leq t \leq T_2.$$

Now $M(\frac{P_o}{2} + tY) = M(\frac{P_o}{2} - \frac{t}{T_1}\frac{P_o}{2} + \frac{t}{T_1}\frac{P_o}{2} + tY)$

$$\geq (1 - \frac{t}{T_1})\, M(\frac{P_o}{2}) + \frac{t}{T_1}\, M(\frac{P_o}{2} + T_1\, Y)$$

$$M(\frac{P_o}{2} + tY) - M(\frac{P_o}{2}) \geq \frac{t}{T_1}\, [M(\frac{P_o}{2} + T_1\, Y) - M(\frac{P_o}{2})].$$

Therefore,

$$M(\frac{P_o}{2} + tY) - [\frac{P_o}{2} - \frac{1}{2} + tz] \geq \frac{t}{T_1}\, [\frac{P_o}{2} + T_1\, Y) - M(\frac{P_o}{2}) - T_1\, z] + \frac{1}{2}$$

Now by the defintion of the range of t, it follows that

$$\frac{-T_1}{2t} < M(\frac{P_o}{2} + T_1\, Y) - \frac{P_o}{2} - T_1\, z < \frac{T_1}{2t},$$

and therefore, $\frac{t}{T_1}\, [M(\frac{P_o}{2} + T_1\, Y) - \frac{P_o}{2} - T_1\, z] > -\frac{1}{2}$

Hence $M(\frac{P_o}{2} + tY) - [\frac{P_o}{2} - \frac{1}{2} + tz] > 0$ for $0 \leq t < T_2$.

So $(\frac{P_o}{2}, \frac{P_o}{2} - \frac{1}{2}) + t(Y, z) \in \hat{K}$ for $0 < t < T_2$. Q.E.D.

It is interesting that properties A and B used in defining a generalized game are necessary to the above result.

Proposition 4:

If $\hat{K}$ has non-void radial kernel and $M(P_o) > -\infty$, then χ spans V and, for each $X \in \chi$, there exists $\eta^* > 0$ such that $M(P_o - \eta^* X) > -\infty$.

Proof:

Suppose $\hat{K}$ is radial at some (X^o, z^o), and let $Y \varepsilon V$, $X \varepsilon \chi$ be given. Then there exist $t_1, t_2 > 0$ such that

$$(W, z^o) \equiv (X^o + t_1 Y, z^o + t_1 \cdot 0) \varepsilon \hat{K} \quad \text{and}$$

$$(X^o - t_2 X, z^o + t_2 \cdot 0) \varepsilon \hat{K}.$$

Hence $Y = \frac{W - X^o}{t}$, with W and X^o in K. But K is spanned by χ, since $M(X) > -\infty$ for all $X \varepsilon K$. So Y is in the span of χ.

To prove the remaining assertion, we note that X^o in K means $-\infty < M(X^o)$ and $X^o + t_2 X$ in K means there is an $\eta_2^* > 0$ such that $M(P_o - \eta_2^* (X^o + t_2 X)) > -\infty$, by the definition of K. Setting $\eta^* = \eta_2^* t_2$ we have $M(P_o - \eta^* X) \geq M(P_0 - \eta_2^* (X^o + t_2 X)) + \eta_2^* M(X^o) > -\infty$.

Theorem 5:

There exists a linear functional λ on V, feasible for Problem I, such that $\lambda(X_o) = v_o$.

Proof:

By Lemma 2 and Proposition 3, $\hat{K}$ and B are disjoint convex sets such that $\hat{K}$ is radial at some point. By the theorem of the separating hyperplane (see [7], page 22) there is a non-trivial linear functional $F(X, z)$ on $V \times R$ such that

$$\sup_{(X, z) \varepsilon \hat{K}} F(X, z) \leq \inf_{(X, z) \varepsilon B} F(X, z).$$

$F(X, z)$ has the following properties:

(i) $\inf\limits_{(X, z) \in B} F(X, z) \le F(0, 0) = 0$ since $(0, 0) \in B$ and F is linear.

(ii) $\sup\limits_{(X, z) \in \hat{K}} F(X, z) \ge F(\frac{1}{n} X_o, \frac{1}{n}(v_o - 1)) = \frac{1}{n} F(X_o, v_o - 1) \to 0,$

since $(\frac{1}{n} X_o, \frac{1}{n}(v_o - 1)) \in \hat{K}$, where $v_o = M(X_o)$,

(iii) inf = sup = 0, from (i) and (ii).

(iv) $F(P_o, p_o - 1) < 0$. For, let (Y, z) such that $F(Y, z) > 0$.

By proposition 3, for $t > 0$ small enough

$$(\frac{P_o}{2}, \frac{1}{2}(p_o - 1)) + t(Y, z) \text{ is in } \hat{K}.$$

But $F(P_o, p_o - 1) < F(P_o, p_o - 1) + 2tF(Y, z) \le 2F((\frac{P_o}{2}, \frac{p_o}{2} - \frac{1}{2}) + t(Y,z)) \le 0$

(v) $F(X, z) = f(X) + \gamma z$ for f linear on V and some $\gamma \in R$.

(vi) $f(P_o) + \gamma(p_o - 1) < 0 \le f(P_o) + \gamma p_o$ implying $\gamma > 0$.

In order to apply theorem 2 of Fan-Glicksberg-Hoffman [8] we observe that $-M(X) + z$ is a convex system of one inequality on the convex set $K \times R$. Further $f(X) + \gamma z$ is linear and hence concave, and $-M(X) + z < 0$, $(X,z) \in K \Rightarrow f(X) + \gamma z \le 0$. Therefore, the generalized Farkas-Minkowski type theorem of Fan-Glicksberg-Hoffman [8] asserts the existence of $k \ge 0$ such that

$$f(X) + \gamma z < k[-M(X) + z] \quad \text{for all } (X, z) \in K \times R.$$

Since $(0, 1)$ and $(0, -1)$ are in $K \times R$, it follows that

$f(0) + \gamma \leq k[-M(0) + 1$ and therefore $\gamma \leq k$. Similarly

$f(0) - \gamma \leq k[-M(0) - 1]$ implies $-\gamma \leq -k$. Hence $k = \gamma > 0$, and

$\frac{-f(X)}{k} \geq M(X)$ for $X \varepsilon K$. Here we use the fact that $M(0) = 0$, for

otherwise $M(X) = \pm \infty$ for all X, and in particular $M(X_o) = v_o = \infty$,

a contradiction.

For each $X \varepsilon K$, let $\lambda(X) = \frac{-f(X)}{k}$. Since $K \supset \chi$ and χ spans V, this induces a linear functional λ on all of V, with the properties that

(i) $\lambda(X) \geq M(X)$ for $X \varepsilon K$, and hence

(ii) $\lambda(X) \geq M(X) \geq v(X)$ for $X \varepsilon \chi$, so λ is feasible for problem I, and

(iii) $0 \geq \lambda(X_o) - v_o \geq M(X_o) - v_o = 0$, so $\lambda(X_o) = v_o$ Q.E.D.

Later on, we will use the fact that this proof does not rely on $X_o \varepsilon \chi$ except in the implicit assumption that $M(X_o) = v_o > -\infty$, which follows from $X_o \varepsilon \chi$.

<u>Corollary 6</u>:

If $v_o = v(X_o)$, the core is non-empty.

The next proposition gives sufficient conditions that the functional values $\{\lambda(X) | X \varepsilon \chi\}$ constitute a bounded set. While the hypotheses may appear strong, they are satisfied by the examples given earlier.

Proposition 7:

Suppose for a weakly balanced game

i) There is a P_o, with $-\infty < M(P_o) < \infty$, and an $\bar{\eta} > 0$ such that $M(P_o - \bar{\eta}X) > -\infty$ for all $X \in \chi$, and

ii) There is a set π χ such that $v(P) = 0$ for $P \in \pi$ and $M_\pi(X) > -\infty$ for $X \in \chi$. Then $|\lambda(X)| \leq \frac{\lambda(P_o)}{\bar{\eta}}$ for all $X \in \chi$ and λ as found in Theorem 5.

Proof:

Hypothesis (ii) implies that $M_\pi(X) = 0$ for all $X \in \chi$, so since $\pi \subset \chi$, it follows that $M_\chi(X) \geq M_\pi(X) = 0$ for all $X \in \chi$. For any $Y \in V$, $M_\chi(Y) > -\infty$ implies $Y = \sum_{i=1}^{n} \eta_i X_i$ with $\eta_i \geq 0$ and $X_i \in \chi$. Hence $M_\chi(Y) \geq \sum_{i=1}^{n} \eta_i M_\chi(X_i) \geq 0$.

In the proof of proposition 3, we showed that the set $\ddot{K}$ contained an interval on an arbitrary ray (Y, z) from the interior point $(\frac{P_o}{2}, \frac{P_o}{2} - \frac{1}{2})$. Let that ray be (X, 0) for any $X \in \chi$, in which case hypothesis (i) implies that the choice $T_1 = \frac{\bar{\eta}}{2}$ guarantees that $M_\chi(\frac{P_o}{2} \pm \frac{\bar{\eta}}{2} X) > -\infty$ since the expression of X in terms of members of χ is trivial. This means that $\frac{P_o}{2} \pm \frac{\bar{\eta}}{2} X$ is in K, and also that

$$M_\chi(\frac{P_o}{2} + \frac{\bar{\eta}}{2} X) \geq 0.$$

X is also in K by property (ii) of that set. By property (i) of the functional λ,

$$\lambda\left(\frac{P_o}{2} + \frac{\bar{\eta}}{2} X\right) \geq M_X\left(\frac{P_o}{2} + \frac{\bar{\eta}}{2} X\right) \geq 0.$$

But $\lambda(P_o) = \lambda(P_o \pm \bar{\eta} X) \mp \lambda(\bar{\eta} X) \geq \mp \lambda(\bar{\eta} X)$, or

$$|\lambda(X)| \leq \frac{\lambda(P_o)}{\bar{\eta}} .$$ This bound is independent of the choice of $X \in \chi$. Q.E.D.

Hypothesis (i) of this proposition is a "uniform" version of property B in the definition of a generalized game. The set π might be called a slack set, by analogy with finite linear programming. If χ contains a slack set, then the equality signs in the definition of weakly balanced can be changed to $\leq$, where the ordering is that which is induced by the positive cone generated by π. The inequalities $\lambda(P) \geq 0$ which will appear in problem I are equivalent to a requirement that λ be a positive linear functional.

An alternative approach to proposition 7 might be to equip V with a topology, state conditions such that λ will be a continuous linear functional and restrict χ to be a bounded set, in which case (see [7], page 45) the range set $\lambda(\chi)$ will be bounded. In particular, λ will be continuous if χ includes a slack set containing an open set (in fact it need be only a Baire set of second category), by [7] theorem 10.10, since χ is bounded below by 0 on the slack set. Other conditions similar to those of Proposition 7 can be stated in order that λ be continuous. However in applications attention is focused on λ restricted to coalitions, since no interpretation adheres to the remaining elements of V, so Proposition 7 has been dealt with in a topology-free manner.

5. Characterization of Redundancy and the Farkas-Minkowski Property

Some of the inequalities $\lambda(X_\alpha) \geq v(X_\alpha)$, $X_\alpha \in \chi$ may hold automatically for every λ satisfying a system of such inequalities on a subset $\psi \subset \chi$. If this is the case, $\lambda(X_\alpha) \geq v(X_\alpha)$ is said to be redundant with respect to ψ. We will be concerned with the non-trivial case $X_\alpha \notin \psi$. If $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to $\psi = \chi - \{X_\alpha\}$, the coalition X_α can be ignored in determining core membership. Furthermore, a new coalition, X_β, might be sought such that $\lambda(X_\beta) \geq v(X_\beta)$ is redundant with respect to $\psi' = \psi - \{X_\beta\}$. In the finite game case, conditions can be given under which reiteration of this procedure leads to a characterization of core membership in terms of coalitions each consisting of a single player [9]. Weaker results are available in the generalized case.

For a given subset $\psi \subset \chi$, denote the subspace of V generated by ψ as V_ψ. Note that if $X_\alpha \notin V_\psi$, $\lambda(X_\alpha) \geq v(X_\alpha)$ cannot be redundant with respect to ψ. This follows from the fact that in this case λ can be defined as $\lambda = (\lambda_\alpha, \lambda_\gamma)$, where λ_α acts on the one-dimensional subspace spanned by X_α, and λ_γ acts on the subspace spanned by $\chi - \{X_\alpha\}$. But inequalities arising from members of ψ affect only λ_γ. So $\lambda_\alpha(X_\alpha)$ may be made less than $v(X_\alpha)$ for any function v.

It turns out that when $M_\psi(X_\alpha) > -\infty$, necessary conditions can be given that $\lambda(X_\alpha) \geq v(X_\alpha)$ be redundant with respect to ψ. However these conditions are shown sufficient only when ψ has an additional property. Fortunately, any ψ can be enlarged to a subset with this property, as follows:

Let F be a linear mapping from V onto V_ψ such that $F(Y) = Y$ for $Y \varepsilon V_\psi$ (the existence of such an F is shown in [10], page 241). F is called a projection of V onto V_ψ. Let $\psi^F = \{X \varepsilon \chi | X \varepsilon \psi$ or $M_\psi(F(X)) = -\infty\}$. Note that $(\psi^F)^F = \psi^F$, so that it is reasonable to deal with sets such that $\psi = \psi^F$. The discussion of redundancy for the most part will be limited to sets ψ such that $\psi^F = \psi$ for some projection F.

Since $\psi^F \supseteq \psi$, necessary conditions for redundancy with respect to ψ^F are also necessary for redundancy with respect to ψ. Furthermore, if $X_\alpha \notin \psi$ and $M_\psi(X_\alpha) > -\infty$ then $X_\alpha \varepsilon V_\psi$ so $F(X_\alpha) = X_\alpha$ and hence $X_\alpha \notin \psi^F$. Thus the enlargement of ψ to ψ^F does not reduce the question of redundancy to a trivial one.

The results of this section follow from this lemma:

<u>Lemma 8</u> Suppose for a weakly balanced game $(\chi, X_0; v)$, $\psi \subset \chi$ satisfies $\psi^F = \psi$ for some projection F and $X_\psi \varepsilon \chi$ satisfies $M_\psi(X_\alpha) > -\infty$. Then

$\inf\{\lambda(X_\alpha) | \lambda(Y) \geq v(Y),\ Y \varepsilon \psi\} \equiv \lambda^*(X_\alpha) = M_\psi(X_\alpha)$, where λ ranges over all linear functionals on V.

<u>Proof</u>:

The result will follow from the application of Theorem 5 to a game over the subspace V_ψ. Recall that the definition of a generalized game required the existence of an element P_0 in V such that for each $X \varepsilon \chi$, $M_\chi(P_0 - \eta^* X) > -\infty$ for some $\eta^* > 0$. We must demonstrate the existence of an element in V_ψ corresponding to P_o.

Let F be the projection given by the hypothesis. Let $P_o' = F(P_o)$; it will be shown to have the desired property. Given $Y \varepsilon \psi$, since $Y \varepsilon \chi$ we can write $\sum \eta_i X_i + \eta^* Y = P_o$ for some $\eta_i, \eta^* \geq 0$ and $X_i \varepsilon \chi$. Since F is linear, $P_o' = F(P_o) = \sum \eta_i F(X_i) + \eta^* F(Y)$. But since $Y \varepsilon \psi$, $F(Y) = Y$. If $X_i \varepsilon \psi$, $F(X_i) = X_i$ so $M_\psi(F(X_i)) > -\infty$. If $X_i \not\varepsilon \psi$ then $M_\psi(F(X_i)) > -\infty$ since $\psi^F = \psi$, so $M_\psi(P_o' - \eta^* Y) = \sum \eta_i M(F(X_i)) > -\infty$. For any $X_\beta \varepsilon \psi$, this shows that $(\psi, X_\beta; v_\psi)$ is a generalized game, where v_ψ is v restricted to ψ.

Furthermore, $(\psi, X_\beta; v_\psi)$ is weakly balanced. The above paragraph shows that $P_o' = \sum_{i=1}^{n} \eta_i Y_j$ with $\eta_i > 0$, $Y_j \varepsilon \psi \subset \chi$. Let $\eta_i^* > 0$ such that $M_\chi(P_o - \eta_i^* Y_j) > -\infty$, let $\nu_i = \eta_i/\eta_i^*$, $i = 1,\ldots,n$, and let $\mu = (\sum_{i=1}^{n} \nu_i)^{-1} > 0$. If $M_\psi(P_o') = \infty$, then $M_\chi(\mu P_o') = \infty$ since $\psi \subset \chi$. Then

$$M_\chi(P_o) = M_\chi(P_o - \mu P_o' + \mu P_o') = M_\chi(P_o - \mu \sum \nu_i \eta_i^* Y_j + \mu P_o')$$

$$= M_\chi(\mu \sum \nu_i (P_o - \eta_i^* Y_j) + \mu P_o')$$

$$\geq \mu \sum \nu_i M_\chi(P_o - \eta_i^* Y_j) + \mu M_\chi(P_o') \geq \infty,$$

a contradiction, so $M_\psi(P_o') < \infty$ and $(\psi, X_\beta; v_\psi)$ is weakly balanced.

Furthermore, $M_\psi(X_\alpha) > -\infty$ by assumption.

If $X_\alpha \in \psi$, then the above shows $(\psi, X_\alpha; v)$ is a weakly balanced generalized game and by Theorem 5 there exists a linear functional λ with $\lambda(Y) \geq v(Y)$, $Y \in \psi$ and $\lambda(X_\alpha) = M_\psi(X_\alpha)$.

Since Theorem 5 does not depend on $X_\alpha \in \psi$ so long as $M_\psi(X_\alpha) > -\infty$ (see note following Theorem 5) this equation holds in any case. But <u>any</u> λ with $\lambda(Y) \geq v(Y)$, $Y \in \psi$ satisfies

$$\lambda(X_\alpha) = \lambda(\sum \eta_i X_i) = \sum \eta_i \lambda(X_i) \geq \sum \eta_i v(X_i)$$

where $X_\alpha = \sum \eta_i X_i$, $\eta_i \geq 0$, $X_i \in \psi$, so therefore $\inf \lambda(X_\alpha) = M_\psi(X_\alpha)$.

<u>Proposition 9</u>: Characterization of Redundancy)

Let $(\chi, X_0; v)$ be a weakly balanced game and $\psi = \psi^F \subset \chi$.

If $M_\psi(X_\alpha) > -\infty$ for some $X_\alpha \in \chi$, $\psi \in \chi$, $(\chi, X_0; v)$ weakly balanced, then the constraint $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to ψ if and only if $M_\psi(X_\alpha) \geq v(X_\alpha)$.

<u>Proof</u>:

Consider the dual programming problems

I_ψ

$$L_\psi = \inf \lambda(X_\alpha)$$

subject to $\lambda(Y) \geq v(Y)$

all $Y \in \psi$

II_ψ

$$M_\psi(X_\alpha) = \sup \sum_{\beta \in B} \eta_\beta v(Y_\beta)$$

subject to

$$\sum_{\beta \in B} \eta_\beta Y_\beta = X_\alpha$$

$$\eta_\beta \geq 0$$

where B runs over all finite index sets of ψ

Since $M_\psi(X_\alpha) > -\infty$, $M_\psi(X_\alpha) = L_\psi$ by Lemma 8. The conclusion now follows by observing that $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to ψ iff $L_\psi \geq v(X_\alpha)$.

<u>Proposition 10</u>:

Let $(\chi, X_0; v)$ be a weakly balanced game and let $X_\alpha \in \psi = \psi^F - \chi$. Suppose that the set $T = \{Y \in \psi | M_\psi(X_\alpha - \eta^* Y) > -\infty$ for some $\eta^* > 0\}$ is non-empty. Then $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to T if and only if it is redundant with respect to ψ.

Consider the dual problems in Proposition 9, and the following pair

(I_T)

$$L_T \equiv \inf \lambda(X_\alpha) \quad \text{subject to } \lambda(Y) \geq v(Y), \quad Y \in T$$

(II_T)

$$M_T(X_\alpha) \equiv \sup_{B'} \sum_{\beta \in B'} \eta_\beta v(Y_\beta) \quad \text{subject to } \sum_{\beta \in B'} \eta_\beta Y_\beta = X_\alpha$$

where B' ranges over all finite index sets of T.

Since $M_\psi(X_\alpha) > -\infty$, it follows that $M_T(X_\alpha) > -\infty$. Also $M_T(X_\alpha) < \infty$ since $M(P_0) < \infty$ (see the argument of Lemma 8). If $X_\alpha \in T$, then $(T, X_\alpha; v)$ is a generalized game with X_α playing the role of P_0, so Theorem 5 yields $L_T = M_T(X_\alpha)$. But Theorem 5 does not depend on $X_\alpha \in T$, so $L_T = M_T(X_\alpha)$ in any case.

Compare (II_ψ) and (II_T). If η is feasible for II_T its non-zero components can be used to form a feasible solution to II_ψ, since $T \subset \psi$. Conversely, if η is feasible for II_ψ, all components corresponding to members of $\psi - T$ are zero. This follows from the definition of T, since the appearance of a $\eta_i > 0$ with $X_i \in \psi - T$ means that $X_i \in T$. So η is feasible for II_T if and only if η is feasible for I_T, so $L_T = M_T(X_\alpha) = M_\psi = L_\psi$. Hence $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to ψ $\therefore$ $L_\psi \geq v(X_\alpha) \Longleftrightarrow L_T \geq v(X_\alpha)$ $\therefore$ $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to T. Q.E.D.

Note that in the case of examples 1 and 2 that T consists of all characteristic functions in ψ of sets $A \in \sum$ such that $A \supset S_\alpha$, where X_α is the characteristic function of S_α. In the finite case Propositions 9 and 10 reduce to Propositions 10 and 9 of [3], respectively.

The condition that $T \neq \phi$ is essential to proposition 10:

Let $\sum_1$ be the field of all subsets of [0, 1], let χ be the corresponding set of characteristic functions, and let X_α be the characteristic function of the set {0}. Furthermore, define $v(X) = 0$ for all X except $v(X_\alpha) = 1$ and $v(Y_n) = 2$ for Y_n the characteristic function of the closed interval $[0, \frac{1}{2}]$, $n = 1, 2, 3, \ldots,$. If $\bar{\psi} = \psi = \chi - X_\alpha$, then $T = \phi$. Clearly $\lambda(X_\alpha) \geq v(X_\alpha) = 1$ is not redundant with respect to T, but if $\lambda(Y_n) \geq 2$. for all n, $\lambda(X_\alpha) \geq 2 > v(X_\alpha)$ so $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to ψ.

The following example points up the importance of the assumption $M_\psi(X_\alpha) > -\infty$ in the above results.

Let $X_y = \binom{(1+y^2)^{-1}}{-1}$, $X_\alpha = \binom{0}{-1}$, $X_\beta = \binom{0}{1}$ and $\chi = \{X_\alpha, X_\beta, X_y | y \geq 0\}$. If we define $v(X_y) = -\tan^{-1} y + y(1+y^2)^{-1}$, $v(X_\alpha) = -\frac{\pi}{2}$ and $v(X_\beta) = 0$, then $(\chi, X_\alpha; v)$ with $P_0 = \binom{1}{1}$ is a weakly balanced generalized game (see [11] for a detailed discussion of this game in terms of semi infinite programming.) If $\psi = \chi - \{X_\alpha\}$, then $X_\alpha \notin V_\psi$ and $M_\psi(X_\alpha) = \infty$. Nonetheless $\lambda(X_\alpha) \geq v(X_\alpha)$ is redundant with respect to ψ. However if we redefine $v(X_\alpha) > -\frac{\pi}{2}$ then $\lambda(X_\alpha) \geq v(X_\alpha)$ is not redundant with respect to ψ.

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In this paper we define the notion of a "weakly balanced game" under very general conditions involving no topology whatever. Using the M-operator, we further extend the work of Schmeidler by establishing duality results for a pair of (possibly) infinite dimensional linear programming problems arising from a generalized game. A necessary and sufficient condition is given in order that a separating hyperplane argument can be employed to prove the existence of a candidate core member for a weakly balanced game. This candidate is shown to be in the core if and only if the game is balanced. No use is made of topological ideas, but conditions are given under which the core member takes on values in a bounded set.

Analogous to results in the n-player case, we use the Charnes-Kortanek M-operator to characterize the redundancy of certain coalition values in restricting core membership.

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